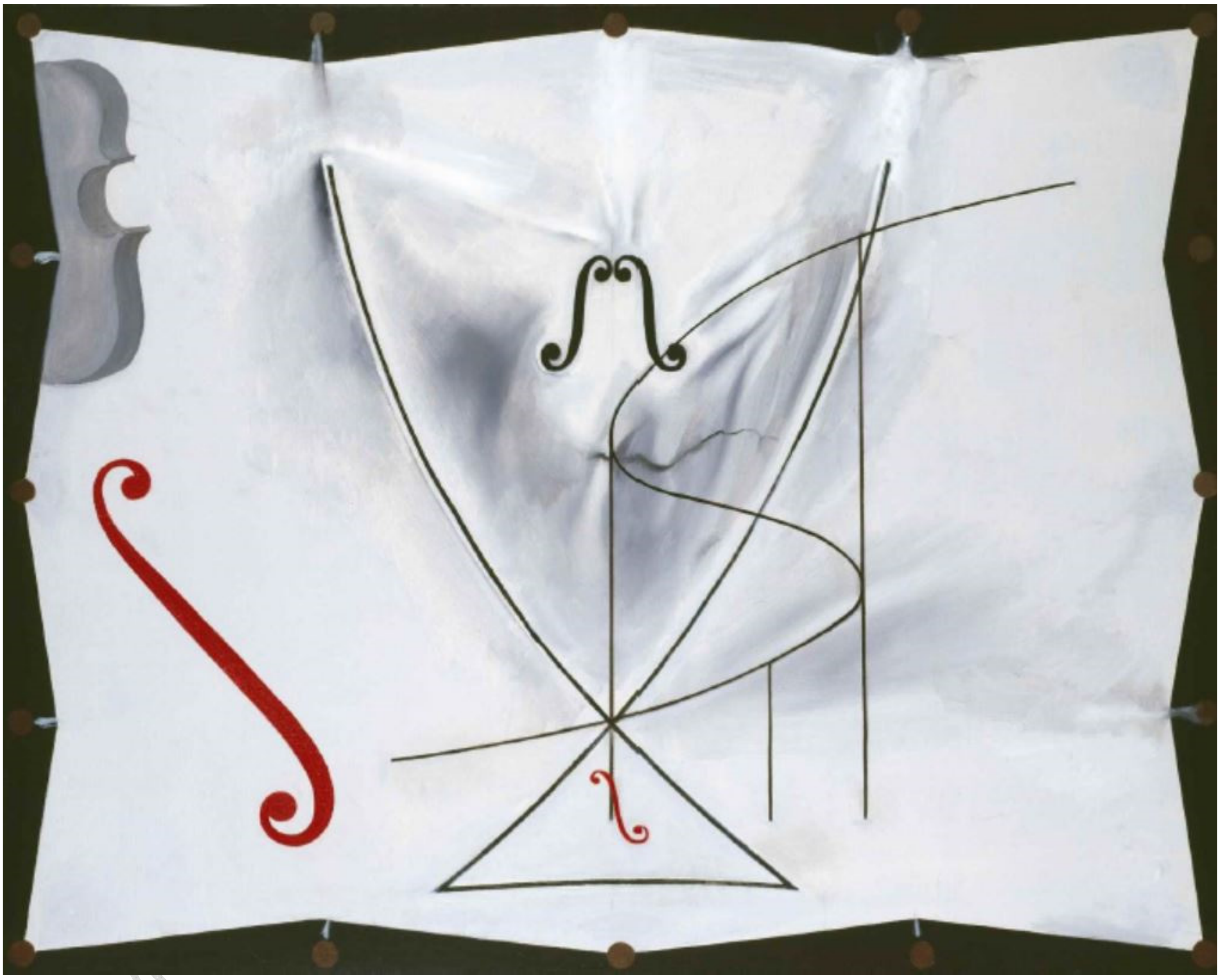
Is DCS a „cusp“ catastrophe?



<u>The frontispiece is:</u>

La Queue d’Aronde - Série des Catastrophes (1983) Salvador Dali

For the following essay I want to thank Herbert Schröder, Dortmund,

who gave me liberally his 2 papers on catastrophe theory:

<u>Schröder, Herbert (31.10.2017) 50 Jahre Katastrophentheorie:</u>

<u>Einführung in die Katastrophentheorie</u>

and:

<u>Bibliographie zur Katastrophenthoerie,</u>

155 (sic!) pages with references

Is DCS a „CUSP“ Catastrophe?

Contents

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Abstract & Preamble

Decompression Sickness (DCS) is the collective name of the pathogenic adverse health effects of excess inert gas load in a human body due to a reduction in ambient pressure. The severity of these effects depends mainly on the gradient and the speed of the pressure reduction. This does not only affect divers or tunnel-/caisson workers but as well aviators and astronauts.

Catastrophe Theory is a part of the mathematical fields of non-linear dynamics and differential topology for the study of bifurcations and singularities of polynomial, non-linear, so-called „potential functions“.

*So the following is an **essay*** on some aspects of DCS and the relation to just *one* of the 7 so-called elementary catastrophes, the cusp. Thus it is not a strict scientific paper, neither in form nor in contents but delineates a couple of preliminary thoughts on the topic, intended to raise awareness or as a basis for further discussions.

As well this should help to unearth again the purely mathematical and useful tools for the qualitative study of catastrophic behavior: these have been buried now for the 2nd. time for ca. 40 years. I want to put it together with a pragmatic, physiologic approach to create an innovative and fresh (and, maybe, a completely unsubstantiated …) new look at dcs-related phenomena.

Rationale

Yet there is no question that Nature fails to be locally linear.

(Source: Mandelbrot, B. B.(1989) Fractal geometry: what is it, and what does it do?, p.4, IBM Yorktown Heights)

With this in mind, we will have a first superficial glance at dcs, non-linear systems and catastrophe theory; or, as Poston & Stewart would have it ([1], p. 1): „The back of a camel is stable, we are told, under a load of N straws, but breaks suddenly under a load of N + 1.“

So a small change in a „control variable“ from N to N+1 results in a sudden jump of a state or „behaviorial variable“, the collapse of the camel's backbone. We could call this rightly so a „catastrophe“, especially if you own this particular camel.

Catastrophe theory deals with the mathematical description of situations where a gradually increase in one stress factor leads, at first, to normally a linear increased response, followed by a sudden jump to a completely different response and different behavior. This is very familiar from a control sample of divers with exactly the same dive (say, from a test-dive in a decompression chamber): the majority has none at all or only sub-clinical symptoms and one is a candidate for a serious dcs treatment. And, as well because:

" … a linear system is precisely equal to the sum of its parts. But many things in nature don't act this way. Whenever parts of a system interfere, or cooperate, or compete, there are nonlinear interactions going on. Most of our everyday life is nonlinear, the principle of superposition fails spectacularly …"

(Source: Steven H. Strogatz (1994): Nonlinear Dynamics and Chaos, Perseus Book, ISBN 0-201-54344-3, p. 9)

Background: What is a „cusp catastrophe“, anyway???

This section is a short primer on catastrophe theory for the mathematically not-so-inclined reader, i.e.: we do not touch on the concepts of: unfolding, codimension, catastrophe germs, perturbations and Maxwell's convention, which are at the rigorous kernel of this theory. These are covered in-depth in [1], [2], [3], [5] & [10]. But for a first, tentative approach with a mapping of dcs to a cusp, these theorems should be clearly the next, but advanced step.

In the mid-1970s with the publication of R. Thom's book [2] and subsequent papers from Zeeman [3], [4] peaked the hype around catastrophe theories, probably due to this epic near-misnomer and the public response in popular journals, extending the topic to the question why a dog starts barking or the price actions at the stock exchange markets. As with Gilmore, who wants to have it in [5] (citation form the appendix: A Brief History on Catastrophe Theory, p. 116):

„This book (i.e.: [2]) was an enigma in both form and substance. It was largely inaccessible to the mathematics community because it was written in the language of biologists, and inaccessible to the biological community due to its presentation of mathematical concepts which seemed to be deep and mysterious.“

But it all really started already at the turn of the century, ca. 1880 with Poincaré and others like Lyapunov, with the consideration of 3-body problems, as an extension of the already by Isaac Newton solved 2-body problem: which turned out to be a mathematical nightmare ([6], p. 209 ff.). These approaches re-surfaced 100 years later and culminated in successful and new insights into many and very different fields, like:

→ Ignition and coherent operation of lasers

→ Collapsing of elastic structures, like bridges

→ Dynamic stability of moving objects and nosediving of airplanes and U-boats

→ Capsizing of naval architectures, like ships or oil-rigs

→ Phase transitions in ferromagnetism and in not-so ideal gases (via the van-der-Waals equation. And that even, as again Poston & Stewart wanted to have it in [1] on p. 327: „The involvement of catastrophe theory with thermodynamics, like most thermodynamic processes, has produced more heat than light.“)

→ Boiling of water; the triple-point (the temperature-/pressure point where a substance is solid, liquid, and gaseous)

→ Breaking of waves

→ the abrupt density changes in a wind tunnel and other, generic shock waves (all these examples in: [7], Chapter 4, p. 95 – 119, and, as well in [3] and [10])

→ Cirrhosis (Ping, Kong (2011): Cusp Bifurcation in the Evolution Model of Cirrhosis, Procedia Environmental Sciences 8 (2011), p. 362 – 370)

→ and, finally the famous heartbeat model by Zeeman ([4], pp. 8 – 67).

They all deal with polynomial. i.e. non-linear energy (potential) functions F. In our case, the candidate function F would be F = F(x, u, v). So our F features one behavior or state variable (x; i.e.: DCS) and 2 control parameters (u & v; for eg.: bottom depth & bottom time of the dive).

To put it a little bit more mathematically: it is basically how to deal with bifurcations and singularities and their approximations through Taylor-Series in order to learn something about this particular function F in the vicinity of local minima, even if this F is not explicitly known. Or, as with Gilmore in his prologue of [10], on p. vi: „Specifically, Thom's Elementary Catastrophe Theory is Poincaré's program applied to the equilibria of dynamical systems which are derivable from a potential function.“

Literature Search

A Quick Search for titles & keywords (29.04.2019) at 7 portals / journals revealed …:

Portal / Journal:	**Keyword DCS / Decompression sickness**	**Keyword catastrophe theory**	**Keyword CUSP catastrophe**	**Keywords DCS & CUSP**	**keywords DCS & catastrophe theory**
1) Science direct (1996 – 2019):					
# hits	3564	64490	193	**0**	**0**
2) Journal of Applied Physiology:					
# hits	864	40	0	**0**	**0**
3) GoogleScholar:					
# hits	5090	422.000	8610	**1 (**)**	**7** (*)
4) Semantic Scholar:					
# hits	ca. 7500	ca. 186.000	ca. 6810	**4** (*)	**0**
5) Federal Science Library, Canada				**0**	**0**
6) Cornell University: arXiv.org	5	90	31	**0**	**0**
7) JSTOR.ORG	19 / 29 (ti) 36 / 427 (ab)	105 (ti) 177 (ab)	5 (ti) 35 (ab)	**0 (ti) 0 (ab)**	**1 (ti)** (*) **1(ab)** (*)

Table 1: Literature Search

(*) PDF with irrelevant contents to our topic here

… more or less: nothing! (ti: title, ab: abstract)

(**) There was only one substantial hit in the:

Handbook of Human Performance, D. M. Jones, A. R. Smith: The Physical Environment, on p. 191, but related to HPNS (High Pressure Nervous Syndrome, colloquially called the “Helium tremor”) and not to DCS (excerpt, yellow display is mine):

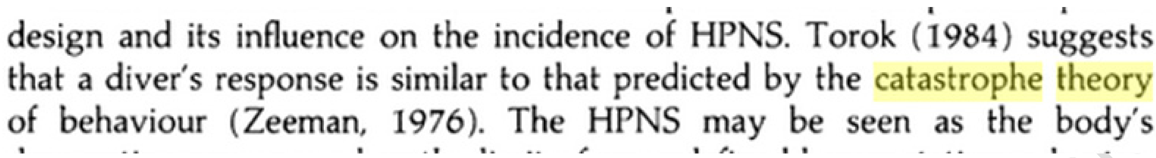

design and its influence on the incidence of HPNS. Torok (1984) suggests that a diver's response is similar to that predicted by the catastrophe theory of behaviour (Zeeman, 1976). The HPNS may be seen as the body's

Figure 1: Example of one substantial hit

Regular search expressions, for example:

"decompression sickness" AND "cusp catastrophe" filetype:pdf gives:

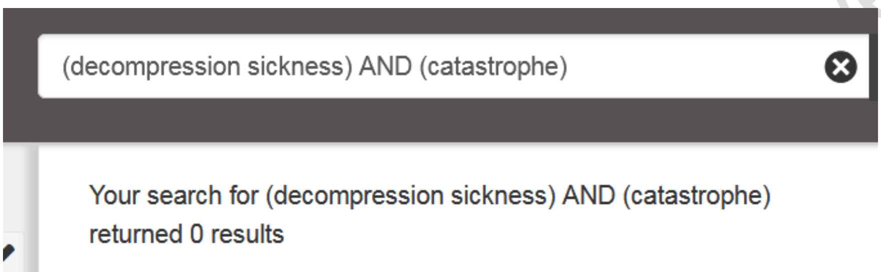


Figure 2: Result with regular search expressions

In fact, simple search with the above basic search expressions via various *popular search engines* revealed just the PDFs from my own lectures and manuals:

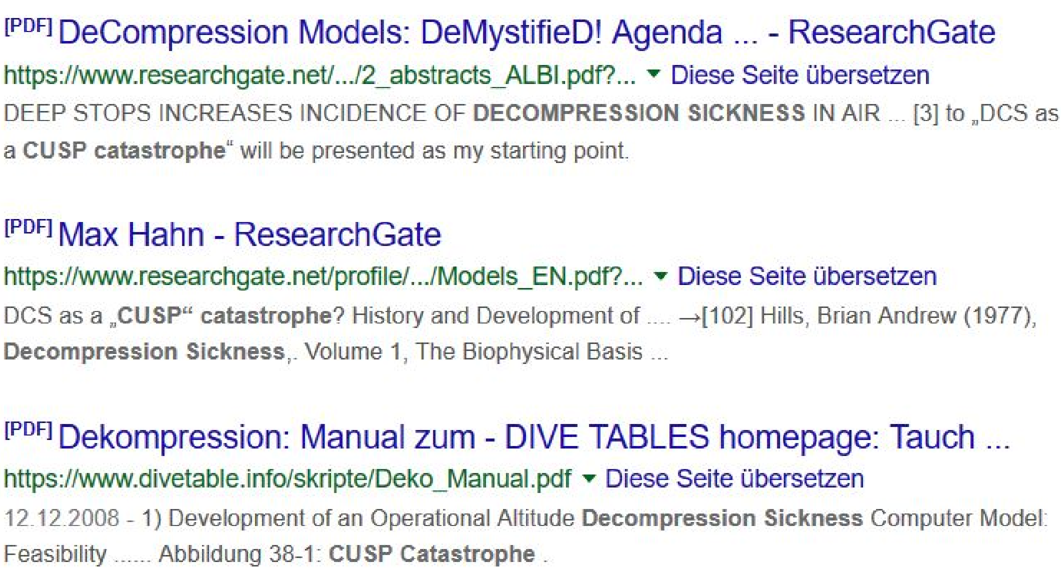


Figure 3: ... more hits!

Basically, i picked the cusp for 4 reasons:

1) It is nicely depictable, especially the „catastrophic behavior“ like a „sudden jump“ („Path 2“ in Fig. 5 or the red arrows in Fig. 8) and the „inaccessible region“ via a simple 3-dimensional graph

2) It is a catastrophe type with 2 control variables which would fit nicely to DCS, say inert gas dose and dehydration, or inert gas dose and workload or workload and skin temperature

3) The cusp resembles very much a dose-response relationship (the classical Hill dose-response equation with the sigmoidal shape), well-known from pharmaco-dynamics

4) The bifurcation set of the equilibrium surface could be mapped to already available dcs data

5) (And yes: well, there is sort of statistical argument in favor of our approach; the majority of the real life examples from the references are just: the cusp).

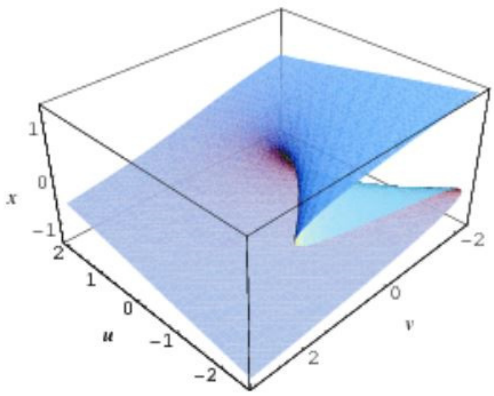
Now let’s examine the first 4 points in closer detail:

Ad 1) the simplest of the 7 catastrophe types, according to Thom’s classification theorem for elementary catastrophes, is the „fold“ with one control variable. This is not enough for a multi-faceted and thus multi-dimensional phenomenon like DCS. The equilibrium surface (blue in Fig. 4) for the cusp is looking like that (thanks to Wolframs MathWorld):

Calculus and Analysis > Differential Geometry > Differential Geometry of Curves > MathWorld Contributors > Sanns >

Cusp Catastrophe

DOWNLOAD Wolfram Notebook



A catastrophe which can occur for two control factors and one behavior axis. The cusp catastrophe is the universal unfolding of the singularity $f(x) = x^4$ and has the equation $F(x, u, v) = x^4 + u x^2 + v x$. The equation $y = x^{2/3}$ also has a cusp catastrophe.

Figure 4: the cusp catastrophe; source: Wolfram Notebook on: http://mathworld.wolfram.com/CuspCatastrophe.html

The behavior or state variable axis (in x-direction: -1 → x → +1 in Fig. 4) would be: „DCS: Yes or No“, or P(DCS), the statistical probability of contracting a decompression sickness; the u- and v- axes are the control variables.

The other elementary catastrophes with bloomy names like „swallow tail“ or „butterfly“ are featuring polynomials of x^5 or higher and 3 or more control variables ([2], p. 65 & p. 69).

Ad 2) the 2 control variables have illustrative names: u is called the normal factor whereas v is called the splitting factor, which will become obvious in Fig. 5. Our definitions could be, for example: u = bottom depth, v = bottom time of the dive, or much better:

bottom depth * bottom time (or, like Hempleman's famous PrT criterion for decompression stress, the product of bottom depth (P) with the squareroot of bottom time (rT)) collapsed into one control variable as an inert gas dose along one axis (say u), thus the 2^{nd}. axis (v) could be one of these: dehydration, workload, skin temperature or age/fitness, for eg.

Ad 3): if you look on the smooth, „un-cusped“ side (left, back of the box in Fig. 4) along, say the axis with u = +1 = const. and you draw a cross-section along this axis, say from v = 3 to v = -3, this cross-section is a stretched and slightly distorted „s“. It is very much like the sigmoidal dose-response curve from a Hill equation, as well seen along „Path 1“ in Fig. 5.

But dose-response means, that *under* the dose-response curve lies hidden a gaussian normal-distribution from our dcs data. That is: the *derivative* of this particular cross-section we drew is basically a normal-distribution, a probability density: the integral of this normal-distribution gives back an accumulated probability density, which is just another word for the dose-response!

This nice result has already been proved with mathematical rigor: that a probability density function of multi-dimensional Gaussian form is a standard-4-parameter cusp and that the first derivative from this particular multivariate Gauss equated with 0 gives the equilibrium surface ([12], p.4 and [13], pp. 311 - 317) shown in Fig. 4.

Ad 4): the bifurcation set is the projection of the blue surface from Fig. 4 onto the u-v-plane along the x-axis, it looks like the dashed curves, the bifurcation lines in Fig. 5.

To accomodate the real DCS data with a standard polynomial cusp catastrophe, we have to scale the axis system because „DCS = No“ or P(DCS) = 0 has to coincide with x = 0. And there may be no negative probabilities.

So if $F(x, u, v) = \frac{1}{4} * x^4 + \frac{1}{2} * u * x^2 + v * x$ (1)

after Thom's original notation (p. 62 & 110 in the english translation, [2a]) for the cusp's potential equation, then the so-called „catastrophe manifold“ or equilibrium surface, i.e. the blue plane in Fig. 4 is defined as:

$\delta F / \delta x = 0$ (that is, we set the 1st. derivative of eq. (1) with respect to x equal to 0) and get:

$$0 = x^3 + u * x + v \quad (2)$$

This is a cubic equation in x, it may have 1 or 3 real solutions, depending on u & v.

The normal to this equilibrium-surface is vertical when $\delta^2F / \delta x^2 = 0$. Thus, by eliminating x via $\delta^2F / \delta x^2 = 0$; that is, we equate as well the 2nd. derivative to 0:

$$0 = 3 * x^2 + u \tag{3}$$

that is: $u = -3 * x^2$ (4)

putting (4) in (2): $v = -x^3 - u * x = -x^3 + 3 * x^2 * x = +2 * x^3$ (5)

and we receive:

$$0 = 4 * u^3 + 27 * v^2 \tag{6}$$

in parametric form of (4), (5): x = t, u = - 3 * t^2, v = 2 * t^3 (7)

as the equation for the bifurcation sets, pls. cf. [1], pp. 78 – 83 & 174 - 176; [2a], p. 62; [3]; [5] & [10], p. 97 - 105.

This was already contemplated by Riemann and Hugoniot as an explanation for the behaviour of a gas-dynamical shock wave from an accelerated piston, that is, a sudden jump in temperature and gas density ([2a], p. 62 - 63).

The vertical re-scaling is done with a new parameter b on the equilibrium surface:

x → x - b

$$0 = (x - b)^3 + u * (x - b) \quad + v \tag{8}$$

As well the 2 control parameters u and v have to be shifted and/or multiplied according to the (dcs-)data: the potential F is dimensionless whereas our control variables are not! Gilmore demonstrates the reduction of physical parameters in [10], p. 204 ff.

There is one caveat to all this stretching & moving: the standard, orthodox catastrophe-polynoms for F insinuate an energy potential, be it kinetic, thermic, chemic or otherwise. The very moment we use bottom time explicitly as a control parameter or implicitly, as in the inertgas dose, the (up to now unknown) *F has lost exactly this functionality*. But this method is already best-practice in social sciences, pls. cf. Courtney Brown in [8].

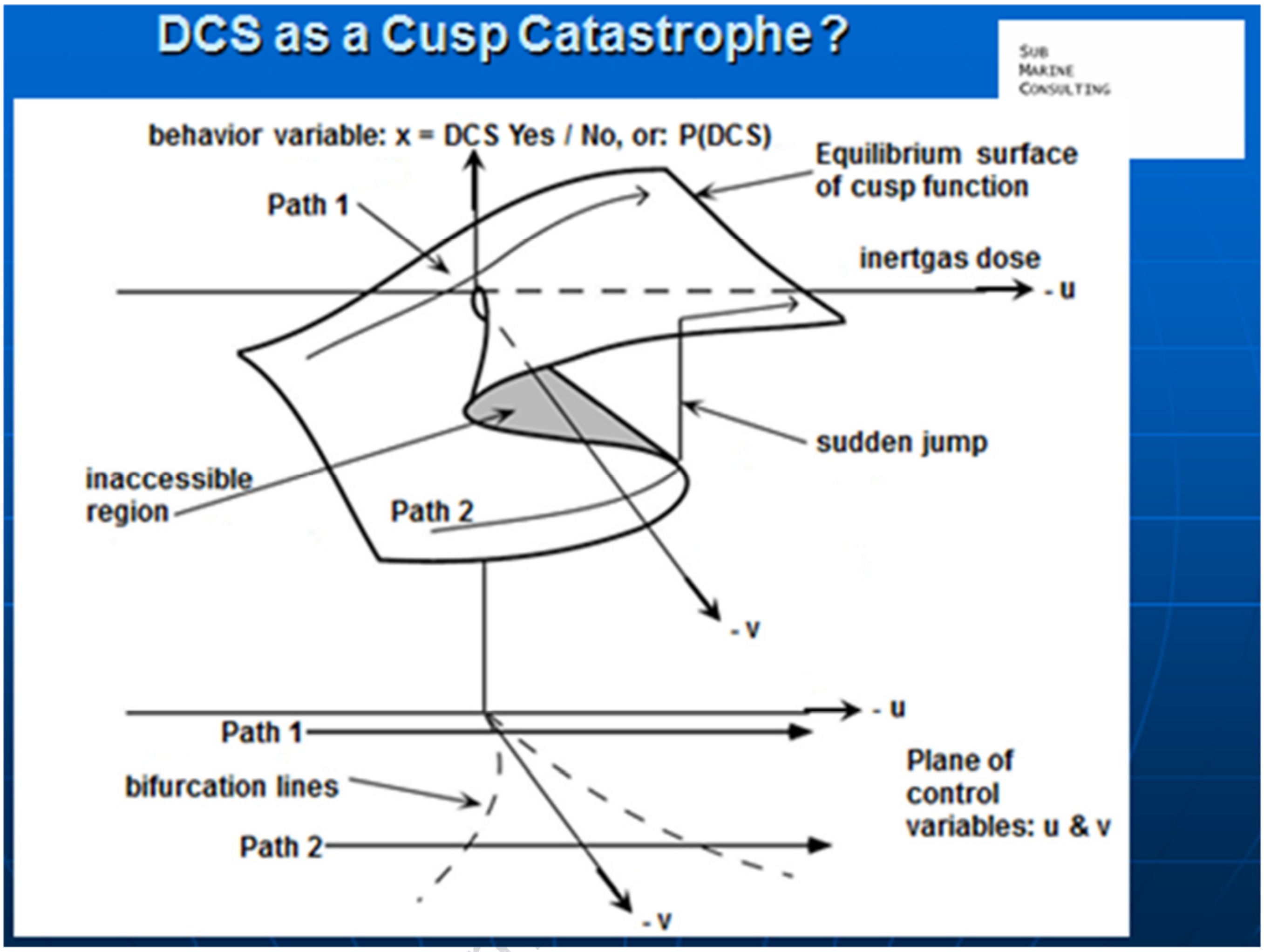


Figure 5: DCS as a cusp, generic

The Bifurcation Set and DCS-data

A *completely different* approach is created by relating the control variables directly with bottom pressure (P) and dive time (T), thus yielding a new „potential“ function F:

$$\text{F(x, u, v)} = \frac{1}{4} * x^4 + \frac{1}{2} * \ P * x^2 + T * x \qquad (1')$$

and by inverting the control plane: if we then could map the traces of dives with dcs, i.e.: the black or white boxes in the vicinity of the red line in Fig. 6, directly to the bifurcation sets is open to conjecture. But it should be determinable if there are more data available.

Fig. 6 is taken from: „Decompression sickness from Commercial Offshore Air-Diving Operations on the UK Continental shelf during 1987“ (HSE OTO 89 029), on p. 22 (the red display along the decompression-stress line of pressure-root time (PrT) of ca. 25 is mine):

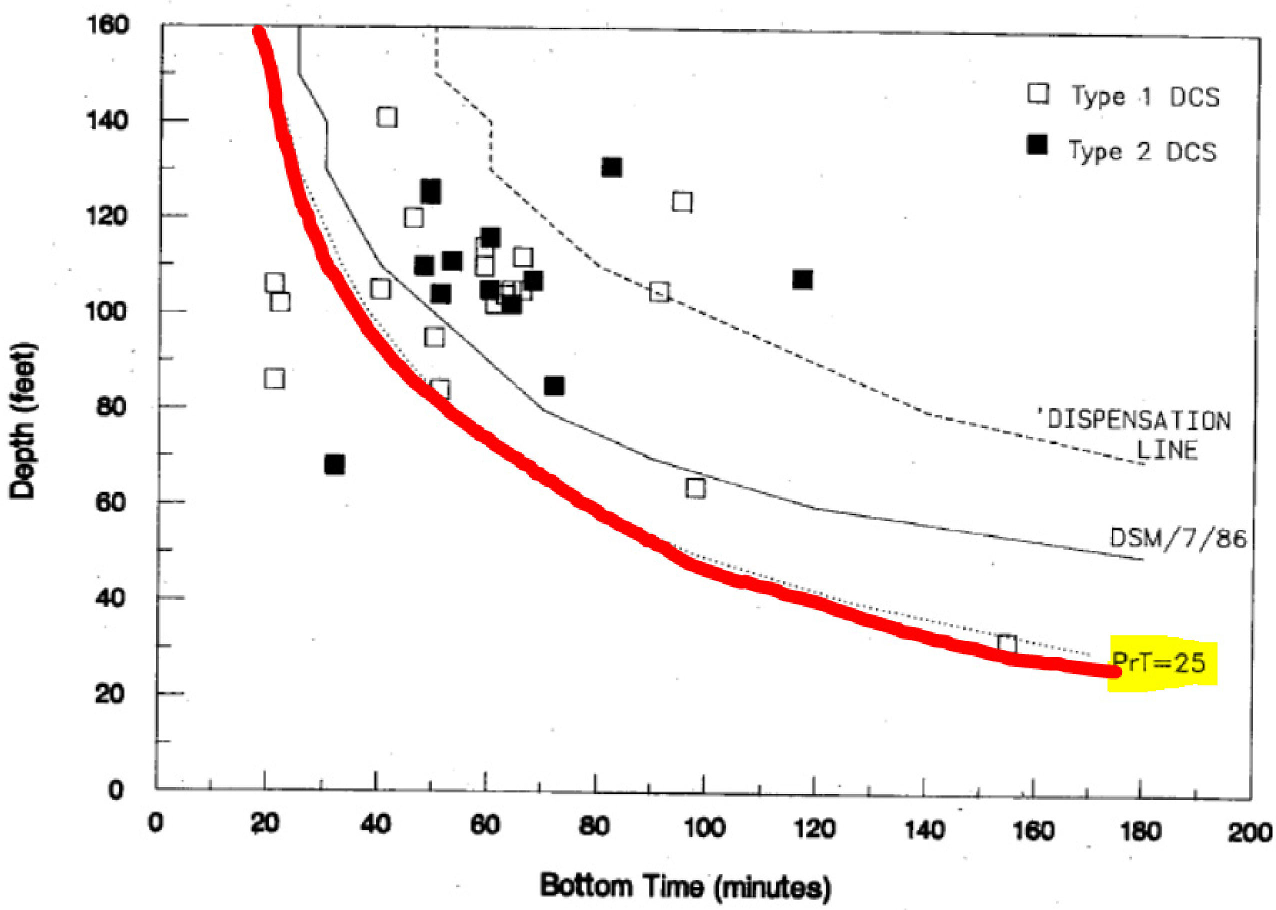


Figure 6: PrT = 25 (Source: HSE OTO 89 029)

Example for PrT: bottom depth = 40 m $\rightarrow$ P = 5 [Bar], dive time T = 36 [min], then

$$PrT = 5 * \text{square root}(36) = 5 * 6 = 30$$

If the decompression stress PrT >= 25 then dcs is more probable, according to this particular data set. Again, the mapping of real dcs data to the generic cusp, which originates around (0,0,0) has to address the above cited 2 points: physical dimension & absolute value. This is done via 2 simple transformations around a desired data region, say a combination of the control variables which feature for eg. a LD_{50} (or P(DCS) = 0.5 or simply the median of our dive profiles) in terms of diving depth P and bottom time T:

Normalization: $P^{\#} = P / P_{50}$ and $T^{\#} = T / T_{50}$

The shifting is done then via analogous to eq. (8):

$$p = P^{\#} - b \text{ and } t = T^{\#} - b$$

(pls. cf. as well [1], p. 328 -330)

To get a grip on the magnitude of this shifting via b we exploit the PrT criterion ($\rightarrow P * T^{1/2} =$ const.) and eliminate one control variable with:

$$P^2 * T = \text{const.}^2 = c^2 \rightarrow T = c^2 / P^2$$

and put that into the bifurcation set (eq. (6)), yielding:

$$0 = 4 * P^3 + 27 * c^4 * P^{-4} \qquad (6')$$

thus preserving the canonical parametrized form (eq. (7)). The new constant $27 * c^4$ could be used to re-determine (or to confirm) the old threshold value of 25. In any case (6‘) gives us an ugly little root if we solve for P:

$$P = \sqrt[7]{\left(-\frac{27}{4}\right) c^4}$$

and we receive many ghastly complex solutions, but one nice real which we may exploit and identify with the direction and magnitude of the required shift: P = - (ca.) 8.266 [Bar].

The 5 catastrophe flags

To decide if a certain behavior displays catastrophic modes, one uses the five, so-called „classical catastrophe flags“, the remaining three could be observed *before* a catastrophe happens ([5], p. 108).

Sudden jumps: the state variable may exhibit a sudden jump with a small variation of any control variable (Path 2 in Fig. 5; red arrows in Fig. 8). Say we get DCS with just a little bit more of dehydration or workload. It is exactly this behavior that makes the cusp attractive: from dives with regularly safe conditions, but already sub-clinical DCS, we get with minor variations a sudden display of full-blown symptoms.

Hysteresis: when the state variable jumps, say from the lower to the upper sheet of the cusp by a certain combination of control parameters, the reverse may not be true, i.e. the jump will happen at other values of the control parameters due to the memory of the system:

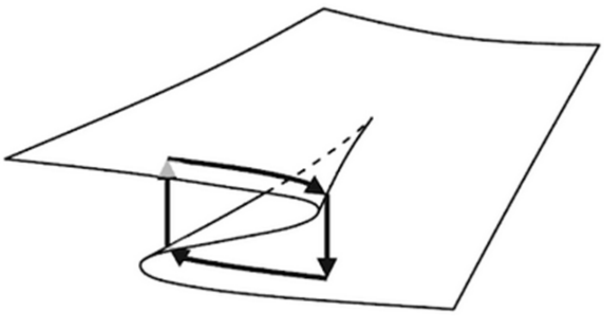


Figure 7: Hysteresis cycle for a cusp catastrophe

As far as a manifested DCS is diagnosed (P(DCS) = +1, the upper sheet of the cusp fold) there will be probably no jump back. So this catastrophe flag will remain relatively uncertain for DCS.

Sensitivity & Divergence: for elementary catastrophes there is sensitivity on some combinations of control variables, i.e.: how robust is the system against perturbations.

Bimodality: in the neighborhood of a catastrophe the system may exhibit 2 or more distinct types of behavior, i.e.: 2 or more values of a scuffproof state variable.

Inaccessibility: the „middle sheet“ of the cusp, pls. cf. the gray-shaded area in Fig.5 (or green in Fig. 8); that is: the state variable may not have observable values from this range. These could be the „marginal“ cases of DCS.

There is a nice graphical overview from Zeeman himself illustrating the 5 flags (Trinity Lecture, 1995):

Figure 8: Qualitative properties of a cusp: the 5 flags (Source: Zeeman 1995)

There are as well more flags to a catastrophic behavior: these are the 3 so-called „diagnostic flags“: divergence of linear response, critical slowing down and anomalous variance ([5], p. 108 – 110; and [10], p. 158 - 183), which could be covered in a next step.

Take-home Messages

- ➔ There are quite interesting aspects which warrant further investigation, provided we could arrange dcs data from many thousands of dives along the 2 control axes of an inert gas dose and with the new parameters, for eg. de-hydration, workload or $VO_{2,max}$ etc.. So, basically:

- ➔ If we manage to map dcs data to the cusp bifurcation set: good for the data! If not, we drop the whole idea or:

- ➔ we try the next, more complex catastrophe type, i.e.: the swallow tail, which looks like that ([2a], p.64):

$$F(x, u, v, w) = \frac{1}{5} * x^5 + \frac{1}{3} * u * x^3 + \frac{1}{2} * v * x^2 + w * x \qquad (9)$$

- ➔ If, then, DCS is no match, then, maybe oxygen toxicity of the central nervous system (CNS-OT). Thom pointed out the action of O_2 on organogenesis ([2a], p. 237 - 241): there as well is a clear cusp catastrophe. Regularly, the onset of CNS-OT and then the sudden development of the seizure could follow the cusp-path: one variable being an O_2 dose (for e.g.: time ^{n} * $pO_2{}^c$), the others CO_2 load, metabolic rate and the like.

- ➔ Be it how it may: Catastrophe Theory helps in understanding that for sudden and dramatic changes in a system no sudden and dramatic changes in the underlying control variables are needed, instead: already very minor, smooth changes could result in gigantic and unexpected systems-behavior, i.e. the sudden jump from silent bubbles to a well-deserved stay in the deco chamber.

In closing, Ilya Prigogine once wrote in his book: “Order out of Chaos”, on p.203:

„The term „counterintuitive“ was introduced at MIT to express our frustration:

„The damn thing just does not do what it should do!“ … … …

We are trained to think in terms of linear causality,

but we need new „tools of thought“:

one of the greatest benefits of models is precisely

to help us discover these tools and learn how to use them.“

Literature: references with comments & further reading

[1] Poston, T. & Stewart, I. N (1978) Catastrophe Theory and its Applications, London, Pitman, ISBN 978-0-486-69271-5

(with a „Bibliography of catastrophe theory“ on 19 pages, plus 227 references)

[2] Thom, René: Stabilité Structurelle et Morphogénèse, W. A. Benjamin, Reading, MA, 1972;

[2a] English translation: Structural Stability and Morphogenesis (by David Fowler), W. A. Benjamin, Reading, MA, 1975.

(There are a lot comments to it, for e.g. two of the most prominent in

- ➔ [9] on p. 89: „ … Zeeman, an ardent admirer of this style, observes that the meaning of Thom's words becomes clear only after inserting 99 lines of your own between every two of Thom's.“
- ➔ Or, from Hermann Haken, my well-nigh doctoral adviser in his famous book „Synergetics“ (1977, Springer) on p. 313: „Thom’s book requires a good deal of mathematical background. Our „pedestrian’s“ approach … entirely different from Thom’s.“)
- ➔ And, finally, by C. H. Waddington in the foreword to the English Edition [2a], p. XV: „And it is difficult mathematics.“

[3] Zeeman, E. Christopher (1977) *Catastrophe Theory: Selected Papers*. Reading, MA: Addison-Wesley.

[4] Zeeman, E.C. (1972) Differential equations for the heartbeat and nerve impulse. In: C.H. Waddington (Ed.) Towards a Theoretical Biology (Vol. 4, pp. 8 – 67), Chicago: Edinburgh University Press

[5] Gilmore, Robert (1992) Catastrophe Theory, 34 pages in "Encyclopedia of Applied Physics, Vol. 3" VCH Publishers

(a pragmatic & concise program overview, brief history, short glossary)

[6] Arnol’d, Vladimir Igorevich (ed.) (1993) Dynamical Systems V: Bifurcation Theory and Catastrophe Theory, Springer, Part II, p. 207 – 271

(a historical overview and 125 further references)

[7] Lu, Yung-Chen (1976) Singularity Theory and an Introduction to Catastrophe Theory, Springer, ISBN 3-540-90221-X

[8] Brown, Courtney (1995) Chaos and Catastrophe Theories; Series: Quantitative Applications in the Social Sciences, Vol. 107, Sage, ISBN 0-8039-5847-1

(a nice little booklet, which covers both these ephemeral related topics with applications from the „soft“ type; and there is even more to it from Courtney:

[8a] Brown, C (1995) Serpents in the Sand: Essays on the Nonlinear Nature of Politics and Human Destiny, The University of Michigan Press

[9] Arnol’d, V. I. (1986) Catastrophe Theory, Springer, ISBN 3-540-16199-6

(108 pages with ca. 70 graphs, only a hand full of formulas, for the laymen in calculus)

[10] Gilmore, Robert (1981) Catastrophe Theory for Scientists and Engineers, Wiley, ISBN 0-471-05064-4

(in-depth, complete, with mathematical rigor, but with infinite many graphs and applications: a genuine treasure chest on 666 pages!)

[11] Saunders, Peter Timothy (1980) An Introduction to Catastrophe Theory, ISBN 0-521-23042-X

[11a] for my german readers: die leicht gekürzte deutsche Übersetzung: Saunders, P. T. (1986) Katastrophentheorie, Eine Einführung für Naturwissenschaftler, Vieweg, ISBN 978-3-528-08939-9

(mit einfachen Beispielen und auch philosophischen Bemerkungen)

[12] Cobb, Loren (April 1980) Estimation Theory for the Cusp Catastrophe Model, REVISED EDITION1, (Online at https://mpra.ub.uni-muenchen.de/37548/MPRA Paper No. 37548) resp. there, more elaborate:

[13] Cobb, Loren; Watson, Bill (1980) Statistical Catastrophe Theory: an Overview, Mathematical Modelling, Vol. 1

Interestingly enough, the book: „On Growth and Form“ by D'Arcy Wentworth Thompson (1917, a new edition 1945 by Macmillan) plays a prominent role in nearly all of the references. So for eg. in [2a] (chpts. 1, 7, 8) or [11] (chpt. 8).

Keywords: dcs, decompression sickness, cusp, catastrophe, catastrophe theory, bifurcation, Riemann-Hugoniot

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Status: DRAFT, # pages: 17 , # words: 4255